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Complete cohomology for arbitrary rings using injectives[1]

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Abstract

We will develop a complete cohomology theory, which vanishes on injectives and give necessary and sufficient conditions for it to be equivalent to the generalized Tate cohomology theory developed by Mislin, Benson and Carlson and Vogel. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently Vogel [9] and Mislin [13] independently developed a generalization of Tate–Farrell cohomology applicable to any group G; in fact, one can define complete cohomological functors for any ring R. Mislin's work was strongly influenced by Gedrich and Gruenberg's theory of terminal completions [8]. In [1] Benson and Carlson give definitions for Tate cohomology for finite groups, but it turned out that they work more generally for any ring R and the resulting theory is isomorphic to that of Vogel and Mislin. A good overview of this complete cohomology theory can be found in [11].

For R-modules M and N these complete cohomology groups are denoted by $\widehat{\operatorname{Ext}}^j_R(M,N)$ and they are defined and can be non-zero for all integers j. They also satisfy the following two properties, where the first one is not surprising as we are considering cohomological functors.

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(1) For every short exact sequence of R-modules there are long exact sequences of complete cohomology with natural connecting homomorphisms in both variables.
(2) Complete cohomology vanishes on projectives in both variables and all dimensions.

This provided the motivation to search for an alternative approach using injectives. It turned out that there is indeed such an injective complete (I-complete) functor $\breve{\mathrm{Ext}}_R^*(-,-)$ satisfying analogous properties. In Sections 2–4 we introduce three approaches to I-complete cohomology, one axiomatic via satellites analogous to Mislin's [13] and the more intuitive approaches analogous to Benson and Carlson's [1] and Vogel's [9]. We shall show that they lead to equivalent functors.

Contrary to ordinary cohomology, where the constructions via projectives and injectives are equivalent, I-completion of $\mathrm{Ext}_R^*(-,-)$ yields a functor not necessarily equivalent to the P-completion. This is connected to Gedrich and Gruenberg's invariants of a ring *silp R*, the supremum of the injective lengths of the projectives, and *spli R*, the supremum of the projective lengths of the injectives [8].

Section 5 will be devoted to proving the following Comparison Theorem.

Theorem 5.2. *Let R be a ring. Then, for all R-modules M and N, the P-complete cohomology $\widehat{\mathrm{Ext}}_R^*(M,N)$ and the I-complete cohomology $\breve{\mathrm{Ext}}_R^*(M,N)$ are naturally equivalent if and only if both silp R and spli R are finite.*

Section 6 will be dedicated to examples. In Section 7 we shall give a brief summary of facts about complete injective resolutions.

2. The approach via satellites

We begin by introducing the necessary notation and give an introduction to basic but important facts about satellites. The main reference for this is [4, Chap. 3].

Let R be an arbitrary ring and M an R-module. Denote by IM the injective envelope of M and by ΣM the cokernel of the inclusion $M \rightarrowtail IM$. We then define inductively, for all $n \geq 1$, $\Sigma^n M = \Sigma(\Sigma^{n-1} M)$ with the convention that $\Sigma^0 M = M$. Let T be a contravariant additive functor from $\mathcal{M}od_R$, the category of R-modules, to $\mathcal{A}b$, the category of abelian groups. We define the left satellite as follows:

$$S^{-1}T(M) = ker(T(\Sigma M) \to T(IM)).$$

The higher satellites are defined inductively by $S^{-n}T(M) = S^{-1}(S^{-n+1}T(M))$ for all $n > 0$, where we put $S^0 T(M) = T(M)$.

Definition 2.1. A family $(T^n | n \in \mathbb{Z})$ of additive functors from $\mathcal{M}od_R$ to $\mathcal{A}b$ is called a contravariant cohomological functor if for each short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of

R-modules, there is a long exact sequence

$$\cdots \xrightarrow{\delta} T^n(A'') \to T^n(A) \to T^n(A') \xrightarrow{\delta} T^{n+1}(A'') \to \cdots$$

with natural connecting homomorphisms δ.

The most well-known example is given by the ordinary Ext-groups $\mathrm{Ext}_R^*(-,N)$ with the convention that $\mathrm{Ext}_R^n(-,N)=0$ for all $n<0$.

Satellites satisfy the following fact, which implies that they are very close to being a contravariant cohomological functor.

Lemma 2.2 (Cartan and Eilenberg [4, (III.3.1)]). *Every short exact sequence of R-modules, $A' \rightarrowtail A \twoheadrightarrow A''$ gives rise to a natural connecting homomorphism $S^{-n}T(A') \to S^{-n+1}T(A'')$, $n>0$, in such a way that the long sequence*

$$\cdots \to S^{-n}T(A'') \to S^{-n}T(A) \to S^{-n}T(A') \to S^{-n+1}T(A'') \to \cdots \to T(A')$$

is exact.

Obviously, for J an injective R-module, we have that $J=IJ$, and therefore $S^{-1}T(J)=0$. Hence, for all R-modules A and all integers $n>k>0$, we obtain a natural isomorphism:

$$S^{-n}T(A) \cong S^{-n+k}T(\Sigma^k A).$$

Note that we call a sequence of contravariant additive functors connected if it satisfies part of Definition 2.1, only requiring that the composite of two consecutive maps in the long sequence is zero. We can now characterize the long exact sequence of satellites in the following way.

Proposition 2.3 (Cartan and Eilenberg [4, (III.5.2′)]). *Let $T^{\leq 0}$ and $U^{\leq 0}$ denote connected sequences of contravariant functors and $\phi^0 : T^0 \to U^0$ a natural transformation. If $U^{\leq 0}$ is a cohomological functor and satisfies $U^{-n}(I)=0$ for all $n>0$ and all injective R-modules I then the following holds:*

(1) *ϕ^0 extends uniquely to $\phi^{\leq 0} : T^{\leq 0} \to U^{\leq 0}$ and $\phi^{\leq 0}$ factors uniquely through the canonical morphism $T^{\leq 0} \to S^{\leq 0}T^0$.*

(2) *If T^0 is half exact and ϕ^0 is an equivalence then the induced morphism $S^{\leq 0}T^0 \to U^{\leq 0}$ is an equivalence.*

There now follow the definition and universal property of the I-completion of a contravariant cohomological functor. This is an analogue to Mislin's definition of the P-completion of a covariant cohomological functor [13, (2.1)].

Definition 2.4. Let (T^*) be a contravariant cohomological functor. Then its I-completion consists of a contravariant cohomological functor $(\check{T}^*)$ together with morphisms $(\tau^*) : (T^*) \to (\check{T}^*)$ satisfying the following conditions:

(1) $\check{T}^n(I)=0$ for all injective R-modules I and every $n \in \mathbb{Z}$.

(2) Every morphism $(T^*) \to (U^*)$, where (U^*) is a contravariant cohomological functor vanishing on injectives, factors uniquely through (τ^*).

Any contravariant cohomological functor satisfying property (1) above will be called I-complete.

Theorem 2.5. *Every contravariant cohomological functor $T^* = (T^n | n \in \mathbb{Z})$ admits a unique I-completion*:

$$\tau^* : T^* \to \check{T}^*$$
$$\check{T}^n(A) = \varinjlim_{k \geq 0} S^{-k} T^{n+k}(A)$$

for all R-modules A and all integers n.

Proof. The proof is exactly the same as in the projective case [13, (2.2)]. We will give a brief outline of the construction here.

For every $n \in \mathbb{Z}$ we obtain a contravariant cohomological functor by defining

$$T^j\langle n \rangle = \begin{cases} S^{j-n} T^n & \text{if } j < n, \\ T^j & \text{if } j \geq n. \end{cases}$$

The identity morphism $T^n \to T^n$ induces a unique morphism $\iota_n^* : T^* \to T^*\langle n \rangle$, where $\iota_n^j = id_{T^j}$ for all $j > n$. In the same fashion, we extend, for all $m > n$, the identity on T^m to a unique morphism $\iota_{n,m}^* : T^*\langle n \rangle \to T^*\langle m \rangle$. Thus, we can now define

$$\check{T}^* = \varinjlim \{T^*\langle n \rangle \mid \iota_{n,m}^*\},$$

which is an I-complete contravariant cohomological functor. We also obtain a natural morphism

$$\iota^* = \varinjlim \iota_n^* : T^* \to \check{T}^*,$$

which satisfies the universal property of Definition 2.4. □

The following two lemmas are analogues to Mislin's Lemmas 2.4 and 2.5 [13] and are proved using similar arguments. They can be useful tools for computations.

Lemma 2.6. *Let T^* be a contravariant cohomological functor and $n_0 \in \mathbb{Z}$ such that $T^n(I) = 0$ for all $n \geq n_0$ and all injective R-modules I. Then $\iota^n : T^n(M) \to \check{T}^n(M)$ is an isomorphism for all $n \geq n_0$ and all R-modules M. In addition, $\check{T}^*$ is naturally equivalent to $T^*\langle n \rangle$.*

Lemma 2.7. *Let $\phi^* : T^* \to V^*$ be a morphism of contravariant cohomological functors with V^* I-complete. Suppose $\phi^n : T^n \to V^n$ is an equivalence for all $n \geq n_0$. Then the induced morphism $\check{T}^* \to V^*$ is an equivalence.*

3. The intuitive approach

The advantage of this approach is that the construction is more intuitive and some applications become much more easily visible.

Let $n \geq 0$ be an integer and define $I_n \operatorname{Hom}_R(M,N)$ to be the set of all R-module homomorphisms $\varphi \in \operatorname{Hom}_R(M,N)$ factoring through a module of injective dimension $\leq n$. We denote the quotient as follows:

$$(M,N)_n = \operatorname{Hom}_R(M,N)/I_n \operatorname{Hom}_R(M,N).$$

We now define categories $I_n \mathcal{M}od_R$ having as objects the R-modules, and morphisms from M to N lying in $(M,N)_n$.

Lemma 3.1. *Σ defines a functor from $I_n \mathcal{M}od_R$ to itself.*

Proof. To begin, we have to show how to obtain a map $\Sigma f : \Sigma A \to \Sigma B$, for a given R-module homomorphism $f : A \to B$. To do so, consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\iota} & IA & \xrightarrow{\pi} & \Sigma A & \longrightarrow & 0 \\
 & & \Big\downarrow f & & \Big\downarrow If & & \Big\downarrow \Sigma f & & \\
0 & \longrightarrow & B & \xrightarrow{\tau} & IB & \xrightarrow{\sigma} & \Sigma B & \longrightarrow & 0
\end{array}$$

As IB is injective we have a well defined map $If : IA \to IB$ such that $If \circ \iota = \tau \circ f$. Thus we have a well defined homomorphism $\Sigma f \in \operatorname{Hom}(\Sigma A, \Sigma B)$ making the diagram commute. Even though it is not uniquely determined in $\operatorname{Hom}(\Sigma A, \Sigma B)$ it is unique in $(\Sigma A, \Sigma B)_n$:

Let $\bar{a} \in \Sigma A$. We then define $\Sigma f(\bar{a}) = \sigma \circ If(a)$ where $a \in IA$ is chosen such that $\pi(a) = \bar{a}$. Suppose that $\hat{I}f$ extends f as well. Then it suffices to show that $(\Sigma - \hat{\Sigma})f$ factors through IB. To do this, define $\Phi : \Sigma A \to IB$ by $\Phi(\bar{a}) = (If - \hat{I}f)(a)$ where $\pi a = \bar{a}$. This is well defined, which can be verified by a routine check. And by its definition we have that $\sigma\Phi = \Sigma f - \hat{\Sigma} f$ which shows that Σf is uniquely determined in $(\Sigma M, \Sigma N)_n$. Verification that Σ is indeed a functor again is a routine check. $\square$

Thus we have shown that, for all integers $n \geq 0$, there is a well defined sequence of maps

$$(M,N)_n \to (\Sigma M, \Sigma N)_n \to (\Sigma^2 M, \Sigma^2 N)_n \to \cdots$$

and it is now possible to define an analogue to the Benson–Carlson groups:

Definition 3.2.

$$\widetilde{BC}{}^0_R(M,N) = \varinjlim_{i \geq 0} (\Sigma^i M, \Sigma^i N)_0.$$

In fact, this definition is independent of the choice of the categories above.

Lemma 3.3. *For all R-modules M and N and all integers $n \geq 0$,*

$$\widetilde{BC}_R^0(M,N) = \varinjlim_{i \geq 0} (\Sigma^i M, \Sigma^i N)_n.$$

Proof. We obviously have, for all R-modules M and N and all integers $n \geq 0$, the following natural surjection:

$$(M,N)_0 \twoheadrightarrow (M,N)_n.$$

The direct limit is an exact functor, hence we have a natural surjection

$$\Phi\colon \varinjlim_{i \geq 0} (\Sigma^i M, \Sigma^i N)_0 \twoheadrightarrow \varinjlim_{i \geq 0} (\Sigma^i M, \Sigma^i N)_n \quad \forall n \in \mathbb{Z}_{\geq 0}.$$

So we only need to show that Φ is injective. Take $x \in (\Sigma^k M, \Sigma^k N)_0$, some $k \in \mathbb{Z}$, which maps to zero under Φ, which means it factors through a module L of finite injective dimension $\leq n$. Since Σ is functorial we know that $\Sigma^n x : \Sigma^{n+k} M \to \Sigma^{n+k} N$ factors through $\Sigma^n L$, an injective. Hence x represents zero in the direct limit. $\square$

Remark. We can make an analogous statement for P-complete cohomology. Denote by $[M,N]_n$ the R-module homomorphisms from M to N which are unique up to homomorphisms factoring through a module of finite projective dimension $\leq n$. Let $\Omega M = ker(FM \twoheadrightarrow M)$, where FM denotes the free R-module on the underlying set of M. Then, for all $n \geq 0$, it follows that

$$\widehat{\mathrm{Ext}}_R^0(M,N) = \varinjlim_{i \geq 0} [\Omega^i M, \Omega^i N]_n.$$

For convenience we shall, from now on denote $(M,N)_0$ by (M,N). Let us further denote, for R-modules M and N, by $[\phi]$ the image of $\phi \in \mathrm{Hom}_R(M,N)$ in (M,N).

Proposition 3.4. *Every short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of R-modules induces a long exact sequence*

$$\cdots \to (\Sigma A, N) \to (\Sigma A', N) \xrightarrow{\delta} (A'', N) \to (A, N) \to (A', N)$$

with natural connecting homomorphism δ.

Proof. From the proof of Lemma 3.1 we can conclude that for an R-module B, every $[\phi] \in (A', B)$ induces a unique $[\psi] \in (A'', \Sigma B)$. Here we take $[\phi] = [id] \in (A', A')$. Thus, $[\psi] \in (A'', \Sigma A')$, which now gives us, for every $[\alpha] \in (\Sigma A', N)$, a unique $[\beta] = [\alpha][\psi] = \delta[\alpha] \in (A'', N)$.

Since $(-, N)$ is half exact, which can be verified with a routine check, we have established exactness at (A, N).

Applying the injective Horseshoe Lemma [4, (I.3.5)] we obtain an exact sequence $(\Sigma A'', N) \to (C, N) \to (\Sigma A', N)$, where $C = coker(A \rightarrowtail IA' \oplus IA'')$. Since, by the injective Schanuel's Lemma [14, Ex. 3.36] $(C,N)\cong(\Sigma A,N)$, we have established exactness at $(\Sigma A,N)$.

Applying the injective Schanuel's Lemma to

$$\begin{array}{ccccc} A' & \rightarrowtail & A & \twoheadrightarrow & A'' \\ \| & & \downarrow & & \downarrow \\ A' & \rightarrowtail & IA' & \twoheadrightarrow & \Sigma A' \end{array}$$

we obtain a short exact sequence

$$A\rightarrowtail A''\oplus IA'\twoheadrightarrow\Sigma A'.$$

Exactness at (A'',N) follows as $(A''\oplus IA',N)\cong(A'',N)$.

To prove exactness at $(\Sigma A',N)$ we use a similar argument. We apply the injective Schanuel's Lemma to the following:

$$\begin{array}{ccccc} A & \rightarrowtail & A''\oplus IA' & \twoheadrightarrow & \Sigma A' \\ \| & & \downarrow & & \downarrow \\ A & \rightarrowtail & IA & \twoheadrightarrow & \Sigma A \end{array}$$

and obtain a short exact sequence $A''\oplus IA'\rightarrowtail IA\oplus\Sigma A'\twoheadrightarrow\Sigma A$. $\square$

Now we shall iterate the process of Proposition 3.4 to obtain a long exact sequence for all $i\in\mathbb{Z}$ and all $j\in\mathbb{N}$:

$$\cdots\to(\Sigma^{i+1}A,\Sigma^jN)\to(\Sigma^{i+1}A',\Sigma^jN)\to(\Sigma^iA'',\Sigma^jN)\to(\Sigma^iA,\Sigma^jN)\to\cdots.$$

This suggests a generalization of Definition 3.2 to all dimensions.

Definition 3.5. Let R be a ring and M and N be R-modules. For every $n\in\mathbb{Z}$ we define

$$\widetilde{BC}_R^n(M,N)=\widetilde{BC}_R^0(M,\Sigma^nN)=\varinjlim_{i\geq|n|}(\Sigma^iM,\Sigma^{i+n}N).$$

Even if Σ^nN is not defined for $n<0$, the definition makes sense, as in the direct limit we only leave out a finite number of initial terms.

If we now take direct limits of the above long exact sequences it turns out that

$$\varinjlim_{n\geq 0}(\Sigma^{i+n}A,\Sigma^{j+n}N)=\widetilde{BC}_R^{-i+j}(A,N).$$

For A' and A we take the same limit. Thus we have shown that the functor defined in Definition 3.5 is a cohomological functor, contravariant in the first and covariant in the second variable, which vanishes on injectives. It now remains to show that, as a functor in the first variable it actually is the I-completion of $\mathrm{Ext}_R^*(-,N)$.

Theorem 3.6. *Let R be a ring and N an R-module. Then, for every integer n, $\widetilde{\operatorname{Ext}}_R^n(-,N)$ and $\widetilde{BC}_R^n(-,N)$ are naturally equivalent.*

Proof. Consider the following injective resolution of N:

$$0 \to N \to IN \to I^2N \to \cdots,$$

where $im(I^jN \to I^{j+1}N) = \Sigma^j N$, for all $j \geq 1$. By dimension shifting in the second variable of the long exact sequence of $\operatorname{Ext}_R^*(M,-)$ we obtain a natural isomorphism

$$\operatorname{Ext}_R^n(M,N) \cong \operatorname{Ext}^1(M, \Sigma^{n-1}N).$$

This implies that there is a natural surjection

$$\bar{\theta}^n : \operatorname{Ext}_R^n(M,N) \twoheadrightarrow (\Sigma^k M, \Sigma^{k+n}N).$$

Passing onto limits we therefore obtain a natural surjection:

$$\Theta^n(M,N) : \varinjlim_{k \geq |n|} \operatorname{Ext}^{n+k}(\Sigma^k M, N) \twoheadrightarrow \varinjlim_{k \geq |n|} (\Sigma^k M, \Sigma^{k+n}N).$$

The connecting maps on the left hand side come from the short exact sequence $\Sigma^k M \rightarrowtail I^{k+1}M \twoheadrightarrow \Sigma^{k+1}M$, for all $k \geq |n|$, and from the corresponding connecting homomorphisms δ in the long exact sequence of $\operatorname{Ext}^*(-,N)$. Hence we have the following equality:

$$\begin{aligned} im\,\delta &= ker(\operatorname{Ext}^{n+k+1}(\Sigma^{k+1}M,N) \to \operatorname{Ext}^{n+k+1}(I^{k+1}M,N)) \\ &= S^{-1}\operatorname{Ext}^{n+k+1}(\Sigma^k M,N) \\ &= S^{-k}\operatorname{Ext}^{n+k+1}(\Sigma M,N), \end{aligned}$$

where the last equality follows from the long exact sequence of Satellites (2.2). Therefore, by the definition of the direct limit we get

$$\begin{aligned} \varinjlim_{k \geq |n|} \operatorname{Ext}^{n+k}(\Sigma^k M,N) &\cong \varinjlim_{k \geq |n|} S^{-k}\operatorname{Ext}^{n+k+1}(\Sigma M,N) \\ &\cong \widetilde{\operatorname{Ext}}^{n+1}(\Sigma M,N) \\ &\cong \widetilde{\operatorname{Ext}}^n(M,N). \end{aligned}$$

It now remains to check that $\Theta^n(M,N)$ is actually injective.

Let $\check{x} \in \widetilde{\operatorname{Ext}}^n(M,N)$ be in the kernel of $\Theta^n(M,N)$. Therefore it can be represented by an element $\bar{x} \in \operatorname{Ext}^{n+k}(\Sigma^k M,N)$, for some $k+n > 0$, whose image in $(\Sigma^k M, \Sigma^{k+n}N)$ is zero. Hence $\Theta^n\bar{x}$ factors through an injective. Using the injective resolution of N as above, we can represent $\bar{x}$ by a cocycle $x : \Sigma^k M \to I^{n+k+1}N$ which factors through $\Sigma^{n+k}N$. Thus we have obtained $y : \Sigma^k M \to \Sigma^{n+k}N$ in the image of $\bar{x}$ in $(\Sigma^k M, \Sigma^{k+n}N)$, which factors through an injective.

From the injective resolution of M, we have the embedding $\iota : \Sigma^k M \rightarrowtail I^{k+1}M$. Hence we can say that y factors through $I^{k+1}M$. We now define a map $\Psi : \Sigma^{k+1}M \to I^{n+k+1}N$ as follows. For every $s \in \Sigma^{k+1}M$ we choose an $i \in I^{n+1}M$ such that $\pi(i) = s$. We then put

$$\Psi : \Sigma^{k+1}M \to I^{n+k}N$$

$$s \mapsto ((Iy) - \tau\phi)(i).$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\Sigma^k M \arrow[r, tail, "\iota"] \arrow[d, "y"] & I^{k+1}M \arrow[r, two heads, "\pi"] \arrow[dl, "\phi"'] \arrow[d, "I_y"] & \Sigma^{k+1}M \arrow[r, equal] \arrow[dl, "\Psi"'] \arrow[d, "\Sigma_y"] & \Sigma^{k+1}M \arrow[d, "z"] \\
\Sigma^{n+k}N \arrow[r, tail, "\tau"] & I^{n+k+1}N \arrow[r, two heads, "\sigma"] & \Sigma^{n+k+1}N \arrow[r, tail, "\tau'"] & I^{n+k+1}N
\end{tikzcd}$$

Ψ is a well defined map such that $\sigma\Psi = \Sigma y$. Now z represents $\delta\bar{x}$ in $\mathrm{Ext}^{n+k+1}(\Sigma^{k+1}M, N)$. Additionally, $z = \tau'\sigma\Psi = d\Psi$ is a coboundary. Thus, in the direct limit we have that $\check{x} = 0$, as required. □

Even though we already know, in particular from the previous theorem, that $\breve{BC}{}_R^*(M,N)$ is an I-complete functor, we are able to state the following much stronger and useful fact, which is an analogue to Theorem 4.2 of Kropholler [10].

Since $\breve{BC}{}_R^*(M,N)$ and $\breve{\mathrm{Ext}}{}_R^*(M,N)$ are naturally equivalent, we will not distinguish between the two notations. From now on we will use the more natural looking $\breve{\mathrm{Ext}}{}_R^*(M,N)$.

Theorem 3.7. *Let M and N be R-modules.*

(1) *If M or N has finite injective dimension then* $\breve{\mathrm{Ext}}{}_R^n(M,N) = 0$, *for all* $n \in \mathbb{Z}$.

(2) $\breve{\mathrm{Ext}}{}_R^0(M,M) = 0$ *if and only if M has finite injective dimension.*

Proof. To prove this theorem we will use the approach to I-complete cohomology we have established in this section. Assertion (1) and the "if"-direction of (2) follow directly from Definition 3.5 and Lemma 3.3. Now suppose

$$\breve{\mathrm{Ext}}{}_R^0(M,M) = \varinjlim_{i \geq 0} (\Sigma^i M, \Sigma^i M) = 0.$$

Then there must be an integer $k > 0$ such that the identity map on M becomes zero in $(\Sigma^k M, \Sigma^k M)$, as otherwise id_M would survive as a non-zero element in the direct limit. Therefore $id_{\Sigma^k M}$ factors through an injective, which means that $\Sigma^k M$ itself is injective.

Thus M has an injective resolution of finite length k. Hence it is of finite injective dimension. □

4. The hypercohomology approach

This is another approach which will make it apparent that I-complete cohomology gives us a cohomological functor in both variables. Additionally it gives us a long exact sequence of cohomological functors involving $\mathrm{Ext}_R^*(-,-)$ and $\widetilde{\mathrm{Ext}}_R^*(-,-)$ analogously to [11, Section 4.4]. We will proceed using the same method laid down by Vogel [9].

Let $M \rightarrowtail \mathbf{I}$ and $N \rightarrowtail \mathbf{J}$ be injective resolutions of the R-modules M and N respectively. Denote by $\mathbf{Hom(M,N)}$ the bicomplex with

$$\mathbf{Hom^n(M,N)} = \prod_{p+q=n} \mathrm{Hom}(I^{-p},J^q), \quad n \in \mathbb{Z}.$$

The boundary map $D_n : \mathbf{Hom^n(M,N)} \to \mathbf{Hom^{n+1}(M,N)}$ is defined as follows. Let $\varphi \in \mathrm{Hom}(I^{-p},J^q)$. Then $D(\varphi) = \delta_*^J \varphi + (-1)^n \varphi \delta_I^*$ where δ_I and δ^J are the boundary maps in $\mathbf{I}$ and $\mathbf{J}$, respectively.

We denote by $\mathbf{Hom_b(M,N)}$ the subcomplex of bounded homomorphisms, which actually is the total complex. An element $f \in \mathbf{Hom^n(M,N)}$ is bounded if there exists a $p_0 \in \mathbb{Z}$ such that $f_p : I^{-p} \to J^q$ is zero for all $|p| > p_0$. Hence

$$\mathbf{Hom_b^n(M,N)} = \prod_{p+q=n} \mathrm{Hom}_b(I^{-p},J^q) = \bigoplus_{p+q=n} \mathrm{Hom}(I^{-p},J^q).$$

Finally denote by $\widetilde{\mathbf{Hom}}\mathbf{(M,N)}$ the quotient complex

$$\widetilde{\mathbf{Hom}}^{\mathbf{n}}\mathbf{(M,N)} = \widetilde{\mathbf{Hom}}^{\mathbf{n}}\mathbf{(M,N)}/\mathbf{Hom_b^n(M,N)}.$$

Passing on to cohomology we define a new functor analogously to Vogel [9]:

Definition 4.1. $V_R^n(M,N) = \mathrm{H}^n(\widetilde{\mathbf{Hom}}\mathbf{(M,N)})$ for all $n \in \mathbb{Z}$.

It can be verified by a routine check analogous to ordinary cohomology that $V_R^n(M,N)$ is independent of the choice of injective resolutions of M and N.

Proposition 4.2. *$V_R^n(M,N)$ is a cohomological functor, contravariant in the first, covariant in the second variable.*

To prove this proposition we need the following lemma.

Lemma 4.3. *Let* $\mathbf{X}$ *be a complex and* $\mathbf{Y}' \rightarrowtail \mathbf{Y} \twoheadrightarrow \mathbf{Y}''$ *a short exact sequence of complexes. If* $\mathbf{Y}'$ *is injective there is a commutative diagram of complexes*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbf{Hom_b}(\mathbf{Y}'',\mathbf{X}) \arrow[r] \arrow[d] & \mathbf{Hom_b}(\mathbf{Y},\mathbf{X}) \arrow[r] \arrow[d] & \mathbf{Hom_b}(\mathbf{Y}',\mathbf{X}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Hom}(\mathbf{Y}'',\mathbf{X}) \arrow[r] \arrow[d] & \mathbf{Hom}(\mathbf{Y},\mathbf{X}) \arrow[r] \arrow[d] & \mathbf{Hom}(\mathbf{Y}',\mathbf{X}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widecheck{\mathbf{Hom}}(\mathbf{Y}'',\mathbf{X}) \arrow[r] \arrow[d] & \widecheck{\mathbf{Hom}}(\mathbf{Y},\mathbf{X}) \arrow[r] \arrow[d] & \widecheck{\mathbf{Hom}}(\mathbf{Y}',\mathbf{X}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows.

Proof. Since $\mathbf{Y}'$ is injective there is, for all $p,q \in \mathbb{Z}$, a short exact sequence of abelian groups

$$0 \to \mathrm{Hom}(Y''^{-p},X^q) \to \mathrm{Hom}(Y^{-p},X^q) \to \mathrm{Hom}(Y'^{-p},X^q) \to 0.$$

Thus, taking direct products gives us exactness of the middle row. Exactness of the top row is a well-known fact, see e.g. [2, Section 5, Proposition 2b]. Hence the bottom row is exact as well. □

We now return to the proof of Proposition 4.2. For each short exact sequence of R-modules $M' \rightarrowtail M \twoheadrightarrow M''$ we obtain a short exact sequence $\mathbf{I}' \rightarrowtail \mathbf{I} \twoheadrightarrow \mathbf{I}''$ of injective resolutions, e.g. [4, (I, 3.5)]. Hence, by the above lemma, there is a short exact sequence

$$0 \to \widecheck{\mathbf{Hom}}(\mathbf{M}'',\mathbf{N}) \to \widecheck{\mathbf{Hom}}(\mathbf{M},\mathbf{N}) \to \widecheck{\mathbf{Hom}}(\mathbf{M}',\mathbf{N}) \to 0$$

of complexes, which, after passing to cohomology, gives us a long exact sequence

$$\cdots \to V^n(M'',N) \to V^n(M,N) \to V^n(M',N) \xrightarrow{\delta} V^{n+1}(M'',N) \to \cdots$$

with natural connecting homomorphism δ, cf. [14, (6.2), (6.3)].

Covariance in the second variable can be derived in the same way as above, from a covariant analogue to Lemma 4.3. □

A family of maps $\varphi=\{\varphi_q \mid q\in\mathbb{Z}\}$ is called an *almost cochain map* of degree n if the following diagram:

$$\begin{array}{ccc} I^q & \xrightarrow{\delta} & I^{q+1} \\ \downarrow{\scriptstyle\varphi_q} & & \downarrow{\scriptstyle\varphi_{q+1}} \\ J^{q+n} & \xrightarrow{\delta} & J^{q+n+1} \end{array}$$

commutes for all but a finite number of $q\in\mathbb{Z}$. Two almost cochain maps $\varphi,\xi:\mathbf{I}\to\mathbf{J}$ of degree n are *almost chain homotopy equivalent* if there exists a family $\psi=\{\psi_i \mid i\in\mathbb{Z}\}$ of R-module homomorphisms $\psi_i:I^{i+1}\to J^{i+n}$ such that $\delta\psi_{i-1}+\psi_i\delta=\varphi_i-\xi_i$ for all but a finite number of $i\in\mathbb{Z}$.

By a standart argument, analogously to e.g. [12, Section 2.3], we can verify that $V^n(M,N)$ is the abelian group of almost cochain homotopy equivalence classes of almost cochain maps.

Theorem 4.4. $V_R^*(-,-)\cong\widetilde{\operatorname{Ext}}_R^*(-,-)$.

Proof. To prove this theorem we will use the approach to I-complete cohomology laid down in Section 3. By the above argument it suffices to show that for arbitrary R-modules M and N there is an isomorphism from the group of almost cochain homotopy equivalence classes of almost cochain maps to $\widetilde{\operatorname{Ext}}_R^*(M,N)$.

Let $[\varphi]$ be a cochain homotopy equivalence class of degree n. Hence, for all $q\geq q_0$, some "big enough" q_0, all the above diagrams commute. Consider the following diagram:

$$\begin{array}{ccccccc} \Sigma^q M & \rightarrowtail & I^q M & \xrightarrow{\delta} & I^{q+1}M & \xrightarrow{\delta} & \cdots \\ \downarrow{\scriptstyle\tilde{\varphi}_q} & & \downarrow{\scriptstyle\varphi_q} & & \downarrow{\scriptstyle\varphi_{q+1}} & & \\ \Sigma^{q+n}N & \rightarrowtail & I^{q+n}N & \xrightarrow{\delta} & I^{q+n+1}N & \xrightarrow{\delta} & \cdots \end{array}$$

Hence φ_q induces a unique $\tilde{\varphi}_q\in\operatorname{Hom}(\Sigma^q M,\Sigma^{q+n}N)$. Suppose, φ and ξ are almost chain homotopy equivalent, i.e. for all $q\geq q_1$, some q_1, there are chain homotopies as above. Thus, $\tilde{\varphi}_q-\tilde{\xi}_q$ factors through an injective, and hence φ_q induces a unique element in $(\Sigma^q M,\Sigma^{q+n}N)$. Therefore $[\varphi]$ gives rise to a unique element in $\widetilde{\operatorname{Ext}}^n(M,N)$.

Now let $\check{\varphi}\in\widetilde{\operatorname{Ext}}^n(M,N)$. It is represented by an element $\bar{\varphi}\in(\Sigma^q M,\Sigma^{q+n}N)$, some $q\in\mathbb{Z}$. Since $\bar{\varphi}=\varphi+I\operatorname{Hom}(\Sigma^q M,\Sigma^{q+n}N)$ for some $\varphi\in\operatorname{Hom}(\Sigma^q M,\Sigma^{q+n}N)$, it gives rise to an almost cochain map. Therefore it suffices to show, that for φ factoring through an injective, it is cochain homotopy equivalent to the zero-map. This chain homotopy map can be constructed in a standard way. □

We shall now proceed to construct the long exact sequence of cohomological functors.

Proposition 4.5. *Let* $\mathbf{D}$ *be a coresolution of* M. *Then, for all integers* $n \geq 0$,

$$\mathrm{H}^n(\mathbf{Hom}(\mathbf{D},\mathbf{J})) \cong \mathrm{H}^n(\mathbf{Hom}(M,\mathbf{J})) = \mathrm{Ext}^n(M,N).$$

Proof. Let $\mathbf{Y}$ be any complex. We denote by $\mathbf{Y}[-1]$ the complex where $Y[-1]_n = Y_{n+1}$. Let $\mathbf{D}'$ be the complex

$$\cdots \to 0 \to M \to D^0 \to D^1 \to \cdots$$

This gives rise to a short exact sequence of complexes

$$\mathbf{D} \rightarrowtail \mathbf{D}' \twoheadrightarrow \mathbf{M}[-1].$$

Since $\mathbf{J}$ is an injective complex the sequence

$$\mathbf{Hom}(\mathbf{M}[-1],\mathbf{J}) \rightarrowtail \mathbf{Hom}(\mathbf{D}',\mathbf{J}) \twoheadrightarrow \mathbf{Hom}(\mathbf{D},\mathbf{J})$$

is still exact. Hence, as $\mathrm{H}^{n-1}(\mathbf{Hom}(\mathbf{M}[-1],\mathbf{J})) = \mathrm{H}^n(\mathbf{Hom}(\mathbf{M},\mathbf{J}))$, it suffices to prove that $\mathbf{Hom}(\mathbf{D}',\mathbf{J})$ is acyclic.

We now denote by $\mathbf{J^i}$ the truncated complex $\cdots \to 0 \to J^0 \to \cdots \to J^i \to 0$. Therefore

$$\mathbf{J} = \varprojlim \mathbf{J^i}$$

is the inverse limit of the complexes $\mathbf{J^i}$. Now, for all $n \geq 0$

$$\mathrm{H}^n(\mathbf{Hom}(\mathbf{D}',\mathbf{J}^*)) = \mathrm{H}^n\left(\mathbf{Hom}\left(D', \varprojlim J^i\right)\right)$$

$$= \mathrm{H}^n\left(\varprojlim(\mathbf{Hom}(\mathbf{D}',\mathbf{J^i})) \rightarrowtail \varprojlim \mathrm{H}^n(\mathbf{Hom}(\mathbf{D}',\mathbf{J^i}))\right).$$

Thus it is sufficient to show that $\mathbf{Hom}(\mathbf{D}',\mathbf{J^i})$ is acyclic. It is the total complex of a third quadrant bicomplex $\mathbf{X}$, $X_{-p,-q} = \mathrm{Hom}(I^p, J^q)$ with exact columns. The spectral sequence

$$H_p H_q(X) \Rightarrow H_{p+q}(\mathbf{Tot}(\mathbf{X}))$$

now gives us that $\mathbf{Tot}(\mathbf{X})$ is acyclic as required. □

Corollary 4.6. *For every integer* $n \geq 0$ *there is a long exact sequence of cohomological functors*

$$\cdots \to X^n \to \mathrm{Ext}^n(M,N) \to \widecheck{\mathrm{Ext}}^n(M,N) \to X^{n+1} \to \cdots$$

where X^n *is the nth cohomology group of the total complex* $\mathbf{Hom_b}(\mathbf{M},\mathbf{N})$.

5. Comparison of I-complete and P-complete cohomology

The aim of this section is to obtain necessary and sufficient conditions for the two cohomological functors $\widecheck{\mathrm{Ext}}_R^*(-,-)$ and $\widehat{\mathrm{Ext}}_R^*(-,-)$ to be naturally equivalent. Even though they are both functors in both variables we only have universal properties for them as functors of one variable. Recall, [14, (2.2)], that $\widehat{\mathrm{Ext}}_R^*(M,-)$ is the P-completion of $\mathrm{Ext}_R^*(M,-)$ as a functor of the second variable, and Theorem 2.5, that $\widecheck{\mathrm{Ext}}_R^*(-,N)$ is the I-completion of $\mathrm{Ext}_R^*(-,N)$ as a functor of the first variable. As a first step, we will create functors $\widecheck{\widehat{\mathrm{Ext}}}_R^*(-,-)$ and $\widehat{\widecheck{\mathrm{Ext}}}_R^*(-,-)$. These are the I-completion of the P-completion and the P-completion of the I-completion respectively. Both functors are now P-complete as well as I-complete and satisfy both universal properties [14, (2.1)] and Definition 2.4. The following proposition is the crucial step for finding sufficient conditions for $\widecheck{\mathrm{Ext}}_R^*(-,-)$ and $\widehat{\mathrm{Ext}}_R^*(-,-)$ to be naturally equivalent.

Proposition 5.1. *Let R be a ring. Then for all R-modules M and N the functors*

$$\widehat{\widecheck{\mathrm{Ext}}}_R^*(M,N) \cong \widecheck{\widehat{\mathrm{Ext}}}_R^*(M,N)$$

are naturally equivalent.

Proof. Denote, for all integers $n \geq 0$, by

$$S_{(1)}^{-1}\mathrm{Ext}_R^n(M,N) = ker(\mathrm{Ext}_R^n(\Sigma M,N) \to \mathrm{Ext}_R^n(IM,N))$$

the left satellite of $\mathrm{Ext}_R^n(-,N)$ regarded as a functor of the first variable, and by

$$S_{(2)}^{-1}\mathrm{Ext}_R^n(M,N) = ker(\mathrm{Ext}_R^n(M,\Omega N) \to \mathrm{Ext}_R^n(M,FN))$$

the left satellite of $\mathrm{Ext}_R^n(M,-)$ regarded as a functor of the second variable. From the definitions of P-complete and I-complete cohomology we obtain for all $n \in \mathbb{Z}$ the following sequences of maps:

$$\mathrm{Ext}^n(M,N) \xrightarrow{d_2} S_{(2)}^{-1}\mathrm{Ext}^{n+1}(M,N) \xrightarrow{d_2} S_{(2)}^{-2}\mathrm{Ext}^{n+2}(M,N) \xrightarrow{d_2} \cdots$$

and

$$\mathrm{Ext}^n(M,N) \xrightarrow{d_1} S_{(1)}^{-1}\mathrm{Ext}^{n+1}(M,N) \xrightarrow{d_1} S_{(1)}^{-2}\mathrm{Ext}^{n+2}(M,N) \xrightarrow{d_1} \cdots$$

Let us now consider the following diagram. For negative n we have to start after a finite number of steps, which will not alter the direct limit.

Diagram 1:

$$\begin{tikzcd}
\mathrm{Ext}^n(M,N) \arrow[r, "\delta_1"] \arrow[d, "\delta_2"] & S^{-1}_{(1)}\mathrm{Ext}^{n+1}(M,N) \arrow[r, "\delta_1"] \arrow[d, "\delta_2"] & S^{-2}_{(1)}\mathrm{Ext}^{n+2}(M,N) \arrow[r] \arrow[d, "\delta_2"] & {} \\
S^{-1}_{(2)}\mathrm{Ext}^{n+1}(M,N) \arrow[r, "\delta_1"] \arrow[d, "\delta_2"] & S^{-1}_{(1)}S^{-1}_{(2)}\mathrm{Ext}^{n+2}(M,N) \arrow[r, "\delta_1"] \arrow[d, "\delta_2"] & S^{-2}_{(1)}S^{-1}_{(2)}\mathrm{Ext}^{n+3}(M,N) \arrow[r] \arrow[d, "\delta_2"] & {} \\
S^{-2}_{(2)}\mathrm{Ext}^{n+2}(M,N) \arrow[r, "\delta_1"] \arrow[d, "\delta_2"] & S^{-1}_{(1)}S^{-2}_{(2)}\mathrm{Ext}^{n+3}(M,N) \arrow[r, "\delta_1"] \arrow[d, "\delta_2"] & S^{-2}_{(1)}S^{-2}_{(2)}\mathrm{Ext}^{n+4}(M,N) \arrow[r] \arrow[d, "\delta_2"] & {} \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

The connecting maps involving mixed satellites are obtained inductively using the following crucial natural isomorphism, see e.g. [4, (III.7.1)]:

$$S^{-j}_{(2)}(S^{-i}_{(1)}S^{-k}_{(l)}\mathrm{Ext}^n(M,N)) \cong S^{-i}_{(1)}(S^{-j}_{(2)}S^{-k}_{(l)}\mathrm{Ext}^n(M,N)),$$

for all integers $i,j,k,n \geq 0$ and $l \in \{1,2\}$.

Claim. Each of the squares

$$\begin{tikzcd}
S^{-i}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+j}(M,N) \arrow[r, "\delta_1"] \arrow[d, "\delta_2"'] & S^{-(i+1)}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+1+j}(M,N) \arrow[d, "\delta_2"'] \\
S^{-i}_{(1)}S^{-(j+1)}_{(2)}\mathrm{Ext}^{n+i+j+1}(M,N) \arrow[r, "\delta_1"] & S^{-(i+1)}_{(1)}S^{-(j+1)}_{(2)}\mathrm{Ext}^{n+i+j+2}(M,N)
\end{tikzcd}$$

in Diagram 1 is anticommutative.

Proof. From the way we have obtained the connecting maps δ_1 and δ_2 it follows that for all $i,j \geq 0$ and $n \in \mathbb{Z}$:

$$Im\,\delta_1 \subseteq S^{-i}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+j+1}(\Sigma M,N)$$

and

$$Im\,\delta_2 \subseteq S^{-i}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+j+1}(M,\Omega N).$$

Thus, to show anticommutativity of the above square it suffices to consider the following square:

$$\begin{tikzcd}
S^{-i}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+j}(M,N) \arrow[r] \arrow[d] & S^{-i}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+j+1}(\Sigma M,N) \arrow[d] \\
S^{-i}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+j+1}(M,\Omega N) \arrow[r] & S^{-i}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+j+2}(\Sigma M,\Omega N).
\end{tikzcd}$$

It is well known that for $i=j=0$ the square is anticommutative, see e.g. [14, (11.24)]. Now suppose $j=0$ and $i=1$. Here we shall consider the following cube:

$$\begin{tikzcd}
S^{-1}_{(1)}\mathrm{Ext}^{n+1}(M,N) \arrow[rr] \arrow[dr] \arrow[dd, tail] & & S^{-1}_{(1)}\mathrm{Ext}^{n+2}(\Sigma M,N) \arrow[dr] \arrow[dd, tail] & \\
& S^{-1}_{(1)}\mathrm{Ext}^{n+2}(M,\Omega N) \arrow[rr] \arrow[dd, tail] & & S^{-1}_{(1)}\mathrm{Ext}^{n+3}(\Sigma M,\Omega N) \arrow[dd, tail] \\
\mathrm{Ext}^{n+1}(\Sigma M,N) \arrow[rr] \arrow[dr] & & \mathrm{Ext}^{n+2}(\Sigma^2 M,N) \arrow[dr] & \\
& \mathrm{Ext}^{n+2}(\Sigma M,\Omega N) \arrow[rr] & & \mathrm{Ext}^{n+3}(\Sigma^2 M,\Omega N).
\end{tikzcd}$$

The side squares all commute. The bottom square is anticommutative by the above. Thus, since all the vertical maps are injective, the top square is anticommutative as required. The case $j=1$ and $i=1$ is proved similarly, and the claim follows by induction.

We proceed now to prove the proposition. Consider now the maps $D_1=\delta_1\delta_1$ and $D_2=\delta_2\delta_2$ and the modified Diagram 1 made up from the following squares:

$$\begin{tikzcd}
S^{-i}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+j}(M,N) \arrow[r, "D_1"] \arrow[d, "D_2"'] & S^{-i-2}_{(1)}S^{-j}_{(2)}\mathrm{Ext}^{n+i+j+2}(M,N) \arrow[d, "D_2"'] \\
S^{-i}_{(1)}S^{-j-2}_{(2)}\mathrm{Ext}^{n+i+j+2}(M,N) \arrow[r, "D_1"] & S^{-i-2}_{(1)}S^{-j-2}_{(2)}\mathrm{Ext}^{n+i+j+4}(M,N)
\end{tikzcd}$$

This now is a commutative square.

Since taking out every other term in the direct system does not change the direct limit, we can now conclude, that there are natural isomorphisms for all $n \in \mathbb{Z}$:

$$\begin{aligned}\widetilde{\widehat{\mathrm{Ext}}}^n(M,N) &= \varinjlim_{i \geq |n+j|} S_{(1)}^{-1} \varinjlim_{j \geq |n|} S_{(2)}^{-j} \mathrm{Ext}^{n+i+j}(M,N) \\ &\cong \varinjlim_{j \geq |n+i|} S_{(2)}^{-j} \varinjlim_{i \geq |n|} S_{(1)}^{-i} \mathrm{Ext}^{n+i+j}(M,N) \\ &\cong \widehat{\widetilde{\mathrm{Ext}}}^n(M,N),\end{aligned}$$

thus proving Proposition 5.1. □

We now have all the ingredients to prove our Comparison Theorem. As already mentioned in the introduction the link between I-complete and P-complete cohomology is strongly related to facts about *silp R* and *spli R*.

Theorem 5.2. *Let R be a ring. Then, for all R-modules M and N, the P-complete cohomology* $\widehat{\mathrm{Ext}}^*_R(M,N)$ *and the I-complete cohomology* $\widetilde{\mathrm{Ext}}^*_R(M,N)$ *are naturally equivalent if and only if both silp R and spli R are finite.*

Proof. Let us first assume that $silp\,R = spli\,R = m$. Let P be any projective R-module. $silp\,R = m$ implies, that $inj.dim_R P \leq m$. Hence, by 3.7, $\widetilde{\mathrm{Ext}}^*_R(M,P) = 0$. Thus, $\widetilde{\mathrm{Ext}}^*_R(M,-)$ is a P-complete functor. By the universal property of the P-completion [14, (2.1)] the identity transformation $\widetilde{\mathrm{Ext}}^*_R(M,N) \to \widetilde{\mathrm{Ext}}^*_R(M,N)$ factors uniquely through $\widehat{\widetilde{\mathrm{Ext}}}^*_R(M,N)$. Hence,

$$\widetilde{\mathrm{Ext}}^*_R(M,N) \cong \widehat{\widetilde{\mathrm{Ext}}}^*_R(M,N)$$

is its own P-completion.

Similarly, $spli\,R = m$ implies that

$$\widehat{\mathrm{Ext}}^*_R(M,N) \cong \widetilde{\widehat{\mathrm{Ext}}}^*_R(M,N)$$

is its own I-completion. Application of Proposition 5.1 now gives us the desired natural equivalence.

We now prove the converse. Suppose $\widehat{\mathrm{Ext}}^n_R(M,N) \cong \widetilde{\mathrm{Ext}}^n_R(M,N)$ for all R-modules M and N and all $n \in \mathbb{Z}$. This implies that $\widehat{\mathrm{Ext}}^0_R(M,M) \cong \widetilde{\mathrm{Ext}}^0_R(M,M)$ for all R-modules M. Hence, by [10, (4.2)] and Theorem 3.7 we have the following equivalence:

$$\begin{aligned} pd_R M < \infty &\Leftrightarrow \widehat{\mathrm{Ext}}^0_R(M,M) = 0 \\ &\Leftrightarrow \widetilde{\mathrm{Ext}}^0_R(M,M) = 0 \\ &\Leftrightarrow inj.dim_R\, M < \infty, \end{aligned}$$

which implies that $silp\,R = spli\,R < \infty$. □

6. Examples

In this section we shall give examples for both cases, where the two cohomologies agree and where they do not agree. A large group of examples can be obtained by applying the following result of Cornick and Kropholler.

Theorem 6.1 (Cornick and Kropholler [5, (Theorem C)]). *Let k be a commutative ring of finite global dimension and let G be an $\mathbf{H}\mathfrak{F}$-group. Then $silp(kG) = spli(kG) = \kappa(kG) = fin.dim(kG)$.*

Here, $fin.dim(R)$ denotes the finitistic dimension, which is the supremum of the projective lengths of modules of finite projective dimension. Let k be a commutative ring and G be a group. Let $\mathfrak{X}$ denote the class of kG-modules M such that $pd_{kF}M < \infty$ for all finite subgroups F of G. We denote $\kappa(kG)$ to be the supremum of the projective dimensions of modules in $\mathfrak{X}$.

The next result of [5] now gives us examples, where the two theories agree.

Example 6.2 (*Cornick and Kropholler* [5, (Corollary C)]). Let k be a commutative ring of finite, global dimension and let G be an $\mathbf{H}\mathfrak{F}$-group of type FP_∞. Then all the above invariants are finite.

We come now to our first example of rings where the two theories differ. It is again an application of [5, (Theorem C)].

Example 6.3. Let k be a commutative ring of finite global dimension and let G be a torsion-free $\mathbf{H}\mathfrak{F}$-group of infinite cohomological dimension. Then $silp(kG) = spli(kG) = \infty$.

Proof. In this example we show that $\kappa(kG) = \infty$. Since G is torsion-free the only finite subgroup is the trivial group. Finite global dimension of k implies that every kG-module has finite projective dimension over k bounded by the global dimension of k. In particular, $k \in \mathfrak{X}$. Since $cd_k G = pd_{kG} k = \infty$, it follows that $\kappa(kG) = \infty$. □

Lemma 6.4. *Let R be any ring. Then $fin.dim\, R \leq silp\, R$.*

This Lemma follows directly from the proof of [5, (Theorem. C)]. Thus, finding a ring of infinite finitistic dimension gives us another counterexample. One example for this case are torsion-free abelian groups of infinite rank. Here $fin.dim(\mathbb{Z}G) = \infty$. This example also falls under the previous case.

There now follows an example of a group of type FP_∞ [3], which does not belong to $\mathbf{H}\mathfrak{F}$ [10].

Example 6.5. For the Thompson group $G = \langle x_1, x_2, \ldots \mid x_n^{x_i} = x_{n+1}, \forall i < n \rangle$ $silp(\mathbb{Z}G) = \infty$.

Brown and Geoghegan prove in [3, (1.8)], that G contains a free abelian group of infinite rank. Hence, by the above, $silp\,\mathbb{Z}G = \infty$.

Our last class of examples is connected to complete projective resolutions of an R-module M, the way Cornick and Kropholler defined them [7, (1.1)]. They are acyclic complexes of projective R-modules $\mathbf{P} = (P_*, \delta)$ indexed by the integers such that $\mathbf{P}$ coincides with a projective resolution of M in sufficiently high dimensions and $\mathrm{Hom}_R(\mathbf{P}, Q)$ is acyclic for every projective R-module Q. If a module M has a complete projective resolution then $\widehat{\mathrm{Ext}}^*_R(M,N) \cong \mathrm{H}^*(\mathrm{Hom}_R(\mathbf{P},N)$ [7, (1.2)].

Theorem 6.6 (Cornick and Kropholler [7, (3.10)]). *For a ring R the following are equivalent*:

(1) $silp\,R = spli\,R < \infty$.

(2) *Every R-module has a complete projective resolution.*

It follows from [6, (5.2)] that non-zero R-modules M where $\mathrm{Ext}^n_R(M,P) = 0$ for all projectives P and all intergers $n \geq 0$, do not have a complete projective resolution. There now follows an example of a group ring kG, where the trivial module does not satisfy the hypothesis of Theorem 6.6. Hence it gives rise to non-equal I-complete and P-complete cohomology.

Example 6.7 (*Mislin* [13, (3.2)]). Suppose $G = \mathrm{GL}_n(K)$, where K is a subfield of the algebraic closure of $\mathbb{Q}$. Then for all projective $\mathbb{Z}G$-modules P, $\mathrm{H}^*(G,P) = 0$.

7. Complete injective resolutions

The last examples in the previous section naturally lead to the notion of complete injective resolutions. We shall give a brief introduction without going into great detail since all the proofs are just dual to those of Cornick and Kropholler [7].

Definition 7.1. Let R be a ring and M be an R-module. Then a complete injective resolution of M is an acyclic complex of injective R-modules $\mathbf{I} = (I^*, \delta)$, indexed by the integers, such that

(1) $\mathbf{I}$ coincides with an injective resolution of M in sufficiently high dimensions.

(2) $\mathrm{Hom}_R(J, \mathbf{I})$ is acyclic for all injective modules J.

Lemma 7.2. *Any two complete injective resolutions are chain homotopy equivalent. If $N \rightarrowtail \mathbf{J}$ is an injective resolution of N and $\mathbf{I}$ is a complete injective resolution of N which coincides with $\mathbf{J}$ in sufficiently high dimensions, then there is a chain-map $\phi : \mathbf{J} \to \mathbf{I}$, where $\phi_i = 0$ if $i < 0$.*

The proof is an induction analogous to [7, (2.4)] relying on the fact that for all integers i, j the sequence

$$\mathrm{Hom}_R(J^i, I^{i-1}) \to \mathrm{Hom}_R(J^i, I^i) \to \mathrm{Hom}_R(J^i, I^{i+1})$$

is exact. $\square$

Theorem 7.3. *Let* $\mathbf{I}=(I^*,\delta)$ *be a complete injective resolution of* N *and* M *be an arbitrary* R*-module. Then*

$$\widetilde{\mathrm{Ext}}_R^*(M,N)\cong \mathrm{H}^*(\mathrm{Hom}_R(M,\mathbf{I})).$$

Proof. Suppose $\mathbf{J}=(J^*,d)$ is an injective resolution of N and $\mathbf{I}=(I^*,\delta)$ agrees with $\mathbf{J}$ above dimension n, say. Then there is a chain map $\mathbf{J}\to\mathbf{I}$ inducing a morphism

$$\mathrm{Ext}^*(-,N)\to \mathrm{H}^*(\mathrm{Hom}(-,\mathbf{I}))$$

of contravariant cohomological functors, which is a natural isomorphism above dimension n. Since $\mathrm{Hom}(J,\mathbf{I})$ is acyclic for injective J, the functor $\mathrm{H}^*(\mathrm{Hom}(-,\mathbf{I}))$ is I-complete. An application of Lemma 2.7 now concludes the proof. □

Corollary 7.4. *Suppose* N *has a complete injective resolution. Then*

(1) *If* J *is an injective module, then* $\mathrm{Ext}^i(J,N)=0$ *in sufficiently high dimensions.*

(2) *If* P *is projective, then* $\widetilde{\mathrm{Ext}}_R^*(P,N)=0$.

We shall now give an outline of how to construct complete injective resolutions for modules over group rings kG.

Definition 7.5. Let G be a group, k be a commutative ring of coefficients. Then define

(1) $\mathrm{B}(G,\mathbb{Z})$ to be the set of bounded functions from G to $\mathbb{Z}$ and

(2) $\mathrm{B}(G,k)=\mathrm{B}(G,\mathbb{Z})\otimes_{\mathbb{Z}} k$ to be the k-algebra of functions from G to k which only take finitely many values.

There is a well defined action of G on $\mathrm{B}(G,k)$ defined as follows: $\varphi^g(\bar g)=\varphi(\bar g g^{-1})$ for all $\bar g, g\in G$. For a detailed account of facts about $\mathrm{B}(G,k)$ the reader is referred to [6]. We will only mention the facts necessary to construct our resolution.

Lemma 7.6 (Cornick and Kropholler [7, (3.2),(3.3)]). *Let* G *and* k *be as above. Then*

(i) $\mathrm{B}(G,k)$ *is free as a* k*-module,*

(ii) *there is a* k*-split inclusion* $k\rightarrowtail \mathrm{B}(G,k)$ *of* kG*-modules.*

For simplicity we shall denote by $B=\mathrm{B}(G,k)$ and by $\bar B$ the cokernel of the injection $k\rightarrowtail B$. Clearly $\bar B$ is k-free. We also denote, for kG-modules M, by $hom(B,M)$ the kG-module of k-homomorphisms from B to M.

Theorem 7.7. *Let* kG *be a group ring and* M *be a* kG*-module. Suppose* $inj.dim_{kG}(hom(B,M))<\infty$. *Then* M *has a complete injective resolution.*

Proof. Replacing M with a suitable cokernel in an injective resolution we may assume that $hom(B,M)$ is injective. From Lemma 7.6 it follows that $hom(\bar B,M)\rightarrowtail$

$hom(B,M) \twoheadrightarrow M$ is exact. Also, $hom(B, hom(\bar{B},M))$ is injective. Denote by $hom^i(\bar{B},M) = hom(\bar{B}, hom(\bar{B},\ldots hom(\bar{B},M\ldots))$, i times. Hence by induction $hom(B, hom^i(\bar{B},M))$ is injective for all $i \geq 0$. Thus, we can form a backwards injective resolution of M:

$$\cdots \to hom(B, hom^i(\bar{B},M)) \to \cdots \to hom(B, hom(\bar{B},M)) \to hom(B,M) \twoheadrightarrow M.$$

Splicing this together with an injective resolution

$$M \rightarrowtail I^0 \to I^1 \to \cdots.$$

of M yields a complete injective resolution $\mathbf{I}$.

It remains to check that $\mathrm{Hom}_{kG}(J,\mathbf{I})$ is acyclic for all injective J. Since $J \rightarrowtail J \otimes B$ splits, it is sufficient to show that $\mathrm{Hom}_{kG}(B \otimes J, \mathbf{I})$ is acyclic. Since

$$\mathrm{Hom}_{kG}(B \otimes J, \mathbf{I}) \cong \mathrm{Hom}_{kG}(J, hom(B,\mathbf{I})),$$

this follows from the fact that $\mathbf{I}$ splits under $hom(B,-)$. □

Corollary 7.8 (Shapiro's Lemma). *Let $H \leq G$ be a subgroup and N be a kG-module having a complete injective resolution. Then, for all kH-modules M,*

$$\widetilde{\mathrm{Ext}}^*_{kH}(M,N) \cong \widetilde{\mathrm{Ext}}^*_{kG}(M \otimes_{kH} kG, N).$$

The proof follows from the fact that a complete injective resolution of N over kG can be regarded as a complete injective resolution of N over kH and an application of the Adjoint Isomorphism [14, (2.11)].

Theorem 7.9. *Let R be any ring. Then the following are equivalent:*
(1) $silp\, R = spli\, R < \infty$.
(2) *Every R-module has a complete injective resolution,*

Proof. The implication (1) $\Rightarrow$ (2) follows from [8, Section 4]. $silp\, R < \infty$ implies 7.1(1) and $spli\, R < \infty$ then implies condition (2).

Now suppose every module has a complete injective resolution. In particular every projective P has one. Hence, by Corollary 7.4. $\widetilde{\mathrm{Ext}}^0(P,P) = 0$. Therefore, by Theorem 3.7, every projective has finite injective dimension, thus implying $silp\, R < \infty$. Also by Corollary 7.4, for all injectives J and arbitrary R-modules N there is an integer $n \geq 0$ such that $\mathrm{Ext}^i(J,N) = 0$ for all $i \geq n$. It follows from a standard argument that n can be chosen independently from the choice of N. Hence $spli\, R < \infty$ as required. □

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